Spanning trees with bounded number of branch vertices*

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Abstract. We introduce the following combinatorial optimization problem: Given a connected graph G, find a spanning tree T of G with the smallest number of branch vertices (vertices of degree 3 or more in T). The problem is motivated by new technologies in the realm of optical networks. We investigate algorithmic and combinatorial aspects of the problem.

1 Introduction

The existence of a Hamilton path in a given graph G is a much studied problem, both from the algorithmic and the graph–theoretic point of view. It is known that deciding if such a path exists is an NP-complete problem, even in cubic graphs G [10]. On the other hand, if the graph G satisfies any of a number of density conditions, a Hamilton path is guaranteed to exist. The best known of these density conditions, due to Dirac [6], requires each vertex of G to have a degree of at least $n/2$. (We shall reserve n to always mean the number of vertices of G, and to avoid trivialities we shall always assume that $n \geq 3$.) Other conditions relax the degree constraint somewhat, while requiring at the same time that $K_{1,3}$ (or sometimes $K_{1,4}$) is not an induced subgraph of G. Excluding these subgraphs has the effect of forcing each neighbourhood of a vertex to have many edges, allowing us to guarantee the existence of a Hamilton path with somewhat decreased degree condition.

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There are several natural optimization versions of the Hamilton path problem. For instance, one may want to minimize the number of leaves [15], or minimize the maximum degree, in a spanning tree of G [11, 12, 14, 20, 22]; either of these numbers is equal to two if and only if G has a Hamilton path. The best known optimization problem of this sort is the longest path problem [7, 13, 2]. (G has a Hamilton path if and only if the longest path has n vertices.) It is known that, unless $P = NP$, there is no polynomial time constant ratio approximation algorithm for the longest path problem, even when restricted to cubic graphs G which have a Hamilton path, cf. [2] and [7] where a number of other nonapproximability results are also discussed. In this paper, we introduce another possible optimization problem - minimizing the number of branch vertices in a spanning tree of G.

A *branch vertex* of G is a vertex of degree greater than two. If G is a connected graph, we let $s(G)$ denote the smallest number of branch vertices in any spanning tree of G. Since a spanning tree without branch vertices is a Hamilton path of G, we have $s(G) = 0$ if and only if G admits a Hamilton path. A tree with at most one branch vertex will be called a *spider*. Note that a spider may in fact be a path, i.e., have no branch vertices. Thus a graph G with $s(G) \leq 1$ admits a spanning subgraph that is a spider; we will say that G admits a *spanning spider*. There is an interesting intermediate possibility: We will call a graph G *arachnoid*, if it admits a spanning spider centred at each vertex of G. (A spider with a branch vertex is said to be *centred* at the branch vertex; a spider without branch vertices, i.e., a path, is viewed as centred at any vertex.) It follows from these definitions that a graph G with $s(G) = 0$ (i.e., a graph with a Hamilton path) is arachnoid, and that every arachnoid graph has $s(G) \leq 1$.

Our interest in the problem of minimizing the number of branch vertices arose from a problem in optical networks [19, 25]. The wavelength division multiplexing (WDM) technology of optical communication supports the propagation of multiple laser beams through a single optical fiber, as long as each beam has a different wavelength. A *lightpath* connects two nodes of the network by a sequence of fiber links, with a fixed wavelength. Thus two lightpaths using the same link must use different wavelengths. This situation gives rise to many interesting combinatorial problems, cf. [3, 8].

We consider a different situation, resulting from a new technology allowing a switch to replicate the optical signal by splitting light. *Light–trees* extend the lightpath concept by incorporating optical multicasting capability. Multicast is the ability to transmit information from a single source node to multiple destination nodes. Many bandwidth–intensive applications, such as worldwide web browsing, video conferencing, video on demand services, etc., require multicasting for efficiency purposes. Multicast has been extensively studied in the electronic networking community and has recently received much attention in the optical networking community [23, 24, 28, 30].

The multicast can be supported at the WDM layer by letting WDM switches make copies of data packets directly in the optical domain via light splitting.

Thus a light–tree enables all-optical communication from a source node to a set of destination nodes (including the possibility of the set of destinations consisting of all other nodes) [30]. The switches which correspond to the nodes of degree greater than two have to be able to split light (except for the source of the multicast, which can transmit to any number of neighbours). Each node with splitting capability can forward a number of copies equal to the number of its neighbours, while each other node can support only "drop and continue", that enables the node to receive the data and forward one copy of it; for the rationale behind this assumption see [30, 1]. It should be noticed that optical multicasting (which can be implemented via light–trees) has improved characteristics over electronic multicasting since splitting light is "easier" than copying a packet in an electronic buffer [23].

However, typical optical networks will have a limited number of these more sophisticated switches, and one has to position them in such a way that all possible multicasts can be performed. Thus we are lead to the problem of finding spanning trees with as few branch vertices as possible.

Specifically, let G be the graph whose vertices are the switches of the network, and whose edges are the fiber links. With $s(G)$ light-splitting switches, placed at the branch vertices of an optimal spanning tree, we can perform all possible multicasts. In particular, if $s(G) = 1$, i.e., if G has a spanning spider, we can perform all possible multicasts with just one special switch. If G is a arachnoid graph, no light–splitting switches are needed. (Recall that the source of the multicast can transmit to any number of neighbours.) If $s(G) > 0$, the minimum number of light-splitting switches needed for all possible multicasts in G, is in fact equal to $s(G)$. Indeed, if k vertices of G are allowed to be branch vertices, then multicasting from one of these vertices results in a spanning tree of G with at most k branch vertices, thus $k \geq s(G)$.

In this paper, we investigate the parameter $s(G)$, with emphasis on graphs which admit a spanning spider, or which are arachnoid, in analogy with the study of graphs which admit a Hamilton path. We show that an efficient algorithm to recognize these graphs or a nice characterization for them is unlikely to exist, as the recognition problems are all NP-complete. In fact, we show that $s(G)$ is even hard to approximate. We explore several density conditions, similar to those for Hamilton paths, which are sufficient to give interesting upper bounds on $s(G)$. Finally, we also relate the parameter $s(G)$ to a number of other well studied graph parameters, such as connectivity, independence number, and the length of a longest path.

In dealing with branch vertices, it is helpful to observe that a cut vertex v of a graph G such that $G - v$ has at least three components must be a branch vertex of any spanning tree of G. We will use this observation throughout our arguments.

1.1 Summary of the paper

The rest of the paper is organized as follows. In Section 2 we study the computational complexity of the problems we introduced and show they are all NP-complete. We also give nonapproximability results for $s(G)$. In Section 3 we explore density conditions which imply strong upper bounds on $s(G)$. In Section 4 we relate the parameter $s(G)$ to other important graph parameters; we also show that there is a polynomial time algorithm to find a spanning tree of a connected graph G with at most $\alpha(G) - 2$ branch vertices, where $\alpha(G)$ is the independence number of the graph. Finally, in Section 5 we conclude and give some open problems.

2 Complexity Results

We shall show that all of the problems we have introduced are NP-complete. Since all of them, when appropriately stated, belong to NP, we shall only exhibit the corresponding reductions. Recall that it is NP-complete to decide whether, given a graph G and a vertex v, there exists a Hamilton path in G starting at v [10].

We begin with the case of graphs G that admit a spanning spider, i.e., with $s(G) \leq 1$.

Proposition 1 *It is NP-complete to decide whether a graph G admits a spanning spider.*

Proof. Suppose G is a given graph, and v a given vertex of G. Construct a new graph G' which consists of three copies of G and one additional vertex adjacent to the vertex v of all three copies of G. It is then easy to see that G' has a spanning spider (necessarily centred at the additional vertex), if and only if G admits a Hamilton path starting at v.

More generally, we prove that it is NP-complete to decide whether a given graph admits a spanning tree with at most k branch vertices:

Proposition 2 *Let k be a fixed non-negative integer. It is NP-complete to decide whether a given graph G satisfies $s(G) \leq k$.*

Proof. In view of the preceding result, we may assume that $k \geq 2$. (Recall that $s(G) = 0$ is equivalent to G admitting a Hamilton path, a well known NP-complete problem.) Let again G be a given graph, and v a given vertex of G. This time we construct a graph G' from $2k$ disjoint copies of G and a complete graph on k additional vertices, by making each additional vertex adjacent to the vertex v of its own two copies of G. It is again easy to check that G' admits a spanning tree with at most k branch vertices (necessarily among the vertices of the additional complete graph), if and only if G admits a Hamilton path starting at v.

The problem of recognizing arachnoid graphs is also intractable:

Proposition 3 *It is NP-complete to decide whether a given graph G is arachnoid.*

Proof. We show that the problem of deciding whether a graph has a spanning spider can be reduced to the problem of deciding whether a graph is arachnoid. Given a graph G on n vertices, we construct new graph G' as follows. Take n copies of G, denoted by $G^1, G^2, \ldots, G^n$. Let $\{v_1^i, v_2^i, \ldots, v_n^i\}$ be the vertex set of G^i $(i = 1, 2, \ldots, n)$. The graph G' is constructed from $G^1, G^2, \ldots, G^n$ by identifying all vertices v_j^j $(j = 1, 2, \ldots, n)$ into one vertex called w. The vertex w is a cut-vertex in G', and hence if G' has a spanning spider, then w must be its center. Moreover, it is clear that such a spider exists if and only if G is arachnoid.

We show now that even approximating $s(G)$ seems to be an intractable problem. (More results on nonapproximability can be obtained by the same technique from other results of [2].)

Proposition 4 *Let k be any fixed positive integer. Unless $P = NP$, there is no polynomial time algorithm to check if $s(G) \leq k$, even among cubic graphs with $s(G) = 0$.*

Proof. This will follow from [2], and the following observation. Let $\ell(G)$ denote the maximum length of a path in G. Thus $\ell(G) = n$ if and only if G admits a Hamilton path, that is, $s(G) = 0$. We claim that in any cubic graph G with $s(G) > 0$

$$\ell(G) \geq \frac{2}{2s(G)+1}(n - s(G)). \tag{1}$$

Consider an optimal spanning tree T of G, and denote by S its set of branch vertices. Thus $|S| = s(G)$. The graph induced on $T - S$ is a union of at most $2s(G) + 1$ paths. Indeed, there are at most $s(G) - 1$ paths joining two branch vertices (these are called *trunks* of T), and at most $s(G) + 2$ paths joining a branch vertex to a leaf (these are called *branches* of T). Thus the two longest paths of $T - S$ have at least the number of vertices counted by the right hand side of the above inequality. Since some path of G must contain both these paths of $T - S$, the inequality follows.

It is shown in [2] that there is no polynomial time algorithm guaranteed to find $\ell(G) \geq n/k$, even among cubic graphs with $\ell(G) = n$. It follows from our inequality that (for large enough n) $\ell(G) \geq \frac{1}{s(G)+1}n$, and the proposition follows.

We close this section by noticing that the inequality (1) can be generalized as follows (thus allowing corresponding generalizations of Proposition 4):

$$\ell(G) \geq \frac{2}{s(G)(\Delta - 1) + 1}(n - s(G)),$$

where Δ is the maximum degree of G.

3 Density

Of the several possible density conditions that assure the existence of a Hamilton path [9], we focus on the following result from [18] and [16]:

Theorem 1 [18][16] *Let G be a connected graph that does not contain $K_{1,3}$ as an induced subgraph.*

If each vertex of G has degree at least $(n-2)/3$ (or, more generally, if the sum of the degrees of any three independent vertices is at least $n-2$), then G has a Hamilton path.

We shall prove a similar result for spanning spiders:

Theorem 2 *Let G be a connected graph that does not contain $K_{1,3}$ as an induced subgraph.*

If each vertex of G has degree at least $(n-3)/4$ (or, more generally, if the sum of the degrees of any four independent vertices is at least $n-3$), then G has a spanning spider.

In fact, we have the following general result which treats the existence of a spanning tree with at most k vertices, and includes both the above theorems (corresponding to $k=0$ and $k=1$):

Theorem 3 *Let G be a connected graph that does not contain $K_{1,3}$ as an induced subgraph, and let k be a nonnegative integer.*

If each vertex of G has degree at least $\frac{n-k-2}{k+3}$ (or, more generally, if the sum of the degrees of any $k+3$ independent vertices is at least $n-k-2$), then $s(G) \leq k$.

The proof of Theorem 3 is omitted from this extended abstract, but we offer the following observations.

Remark 1 We first notice that there is no new result of this type for arachnoid graphs, because the smallest lower bound on the degrees of a connected graph G without $K_{1,3}$ which would guarantee that G is arachnoid is $\frac{n-2}{3}$: The graph R_p obtained from a triangle abc by attaching a separate copy of $K_p, p \geq 3$, to each vertex a, b, c, is a $K_{1,3}$-free graph with minimum degree $\frac{n-3}{3}$, which is not arachnoid (all spanning spiders must have center in the triangle abc). However, if all degrees of a connected graph G without $K_{1,3}$ are at least $\frac{n-2}{3}$, then already Theorem 1 implies that G has a Hamilton path (and hence is arachnoid).

We also remark that Theorem 3 does not hold if $K_{1,3}$ is not excluded as an induced subgraph: For $k=0$ this is well-known, and easily seen by considering, say the complete bipartite graph $K_{p,p+2}$ $(p \geq 1)$. For $k \geq 1$ we can take a path on $k+1$ vertices and attach a K_p $(p \geq 2)$ to every vertex of the path. Moreover, we attach an extra K_p to the first and the last vertex of the path. The resulting graph is not $K_{1,3}$-free, and has no spanning tree with k branch vertices. However, the degree sum of any $k+3$ independent vertices in the graph is at least $n-k-1$, where $n = (k+3)p - 2$.

Finally, we note that the bound $n - k - 2$ in the theorem is nearly best possible. For $k = 0$, this is again well-known, and can be seen by considering, say, the above graph R_p. For $k \geq 1$, we consider following example. Take two copies of R_p, where $p = k+1$. Shrink one K_p back to a vertex in one of the two copies of R_p, and attach the vertex to a vertex of degree $p-1$ in the other copy of R_p. The resulting graph has four copies of K_p, each with $p-1$ vertices of degree $p-1$, and one copy K_p, denoted by K, with $p-2$ vertices of degree $p-1$. Take $k-1$ vertices of degree $p-1$ in K, and attach a copy of K_p to each. The resulting graph is $K_{1,3}$-free, and has no spanning tree with at most k branch vertices. The degree sum of any $k+3$ independent vertices is at least $n-k-5$, where $n = (k+3)p + 2$.

We can prove a similar result for graphs that do not contain $K_{1,4}$ as an induced subgraph. (For the existence of Hamilton paths and cycles, such results can be found in [17, 4].)

Naturally, there is a tradeoff between this weaker assumption and the minimum degrees in G one has to assume. We only state it here for $k \geq 1$, the main emphasis of our paper. The proof is omitted from this extended abstract.

Theorem 4 *Let G be a connected graph that does not contain $K_{1,4}$ as an induced subgraph, and let k be a nonnegative integer.*

If each vertex of G has degree at least $\frac{n+3k+1}{k+3}$ (or, more generally, if the sum of the degrees of any $k+3$ independent vertices is at least $n+3k+1$), then $s(G) \leq k$.

We have also attempted to generalize the best known density theorem on hamiltonicity - the theorem of Dirac [6], which implies, in particular, that a graph in which all degrees are at least $\frac{n-1}{2}$ has a Hamilton path. We have the following conjecture:

Conjecture *Let G be a connected graph and k a nonnegative integer.*

If each vertex of G has degree at least $\frac{n-1}{k+2}$ (or, more generally, if the sum of the degrees of any $k+2$ independent vertices is at least $n-1$), then $s(G) \leq k$.

We believe the conjecture, at least for $k = 1$. (Note that it is true for $k = 0$ by Dirac's theorem.) We have proved, for instance, that any connected graph G with all degrees at least $\frac{n-1}{3}$ has either a spanning spider with at most three branches, or has a spanning caterpillar. We are hoping to prove that the spanning caterpillar can be chosen so that all leaves except one are incident with the same vertex, i.e., that the caterpillar is also a spider. This would imply the conjecture for $k = 1$, and would also give additional information on the kind of spiders that are guaranteed to exist - that is, either a spider with at most three branches, or a spider with many branches but all except one having just one edge. We know that having all degrees at least $\frac{n-1}{3}$ is not sufficient to imply the existence of a spider with at most three branches - the graph $K_{p,2p}$ is a counterexample.

4 Relation to Other Problems

We relate our parameter $s(G)$ to other classical graph theoretic parameters.

First we make some additional remarks about arachnoid graphs.

Proposition 5 *If G is an arachnoid graph, then, for any set S of vertices, the graph $G - S$ has at most $|S| + 1$ components.*

Proof. If the deletion of S leaves at least $|S| + 2$ components, then no spider centred in one of the components can be spanning.

The condition in the proposition is a well-known necessary condition for a graph to have a Hamilton path. Recall that we have also observed in Section 3, that we don't have a density condition which implies that a graph is arachnoid without also implying that it has a Hamilton path. Thus we are lead to ask whether or not every arachnoid graph must have a Hamilton path. This is, in fact, not the case, but examples are not easy to find. One can, for instance, take a *hypotraceable graph* G, that is a graph which does not have a Hamilton path, but such that for each vertex v, the graph $G - v$ has a Hamilton path. Hypotraceable graphs are constructed in [26, 27].

Proposition 6 *Every hypotraceable graph is arachnoid.*

Proof. For any vertex x, consider the Hamilton path in $G - v$, where v is any neighbour of x. Adding the edge xv yields a spanning spider of G centred at x.

The following observation shows the relationship between path coverings and $s(G)$.

Proposition 7 *If G is a connected graph whose vertices can be covered by k disjoint paths, then G has a spanning tree with at most $2k - 2$ branch vertices, that is,*

$$s(G) \leq 2k - 2.$$

Let $\alpha(G)$ denote the *independence number* of G, i.e. maximum size of an independent set of vertices in G, and let $\kappa(G)$ denote the *connectivity* of G, i.e. minimum number of vertices removal of which disconnects G or results in an empty graph. Chvátal and Erdős [5] proved that vertices of any graph G can be covered by at most $\lceil \alpha(G)/\kappa(G) \rceil$ vertex disjoint paths. Using the previous proposition we have

Theorem 5 *Let G be a connected graph. Then*

$$s(G) \leq 2 \lceil \alpha(G)/\kappa(G) \rceil - 2.$$

Thus, for 1-connected graphs, previous theorem gives $s(G) \leq 2\alpha(G) - 2$. One may in fact do a little bit better as shown by the following Theorem 6. A caterpillar in which all branch vertices are of degree 3 shows that Theorem 6 is best possible.

Theorem 6 *Let G be a connected graph which is not complete. Then*

$$s(G) \leq \alpha(G) - 2,$$

and a spanning tree with at most $\alpha(G) - 2$ branch vertices can be found in polynomial time.

Proof. We shall prove a more general statement. Let $T \subseteq G$ be a tree and let L be the set of all leaves and S be the set of all branch vertices of T. For every leaf $u \in L$, there is unique branch vertex $s(u) \in S$ (closest to u). These two vertices are joined by unique $s(u) - u$ path in T called the *branch of* u. Recall that the path joining two branch vertices of T is called a trunk of T. Note that a tree with l branch vertices has at least $l+2$ branches (and hence at least $l+2$ leaves).

Suppose that G is a counterexample to our theorem, i.e., it has no spanning tree with at most $\alpha(G) - 2$ branch vertices. Let $T \subseteq G$ be a tree of G with at most $\alpha(G) - 2$ branch vertices. We assume T is chosen so that it contains the greatest number of vertices, and, subject to this condition, so that

(i) the sum of the degrees of the branch vertices of T is as small as possible.

We may assume that T has exactly $\alpha(G)-2$ branch vertices, by the connectivity of G and the maximality of T. It follows that T has at least $\alpha(G)$ leaves. If any two of these, say $u, v \in L$ are adjacent, then the tree $T + uv - us(u)$ violates (i). Thus, the set of leaves of T is an independent set of cardinality $\alpha(G)$. Since G is a counterexample, there is a vertex r in $V(G) - V(T)$. As the maximum independent set in G has cardinality $\alpha(G)$, the vertex r must be adjacent to at least one leaf of T, contradicting the maximality of T.

To see that the above process is constructive, it is enough to start with any tree T and then to modify T so that:

1. It has at most $\alpha(G)-2$ branch vertices. This can be done, since if T has more than $\alpha(G) - 2$ branch vertices, then it has more that $\alpha(G)$ leaves, and at least two of them, say u, v are adjacent. Now, we modify T to $T+uv-us(u)$. This either results in $\alpha(G) - 2$ branch vertices in T, or we can repeat the process.

2. It has at most $\alpha(G)$ leaves. This can be accomplished by a procedure similar to the above.

3. It contains all the vertices of G. Indeed, if T has fewer than $\alpha(G)-2$ branch vertices, then we can simply add any new vertex into T. Otherwise T has at least $\alpha(G)-2$ branch vertices thus, at least $\alpha(G)$ leaves, and any vertex not in T must be adjacent to a leaf. This obviously allows for an extension.

To check which case applies takes polynomial time, and each of the cases is applied at most $|V(G)|$ times.

5 Concluding Remarks

In Section 2 we have proved that, unless $P = NP$, there is no polynomial time algorithm to approximate the minimum number of branch vertices $s(G)$ of any spanning tree of G, to within a constant factor. Under stronger complexity assumptions, stronger nonapproximability results can be derived. The first natural question to ask is which kind of positive results on the approximability of $s(G)$ one can obtain.

In Section 4 we proved that there exists a polynomial time algorithm to find a spanning tree of a graph G with at most $\alpha(G) - 2$ branch vertices, for any connected graph G. A (better) nonconstructive bound is given in Theorem 5 depending on the connectivity $\kappa(G)$ of G. An interesting problem is to improve Theorem 6 constructively taking into account the connectivity $\kappa(G)$ of G.

We have observed that every graph with a Hamilton path (such graphs are sometimes called *traceable*) is arachnoid, and proved in Proposition 6 that every hypotraceable graph is arachnoid, but these are the only ways we have of constructing arachnoid graphs. It would be interesting to find constructions of arachnoid graphs that are neither traceable (do not have a Hamilton path), nor hypotraceable.

In the same vein, all our arachnoid graphs have the property that each vertex is the center of a spider with at most three branches. We propose the problem of constructing arachnoid graphs in which this is not so, i.e., in which some vertex v is the center of only spiders with more than three branches.

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